Some Attempts to Formally Generalise Inverse Semigroups

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April 21, 2004
February 5, 2007

Abstract

In this document results and ideas are given which were found when I tried to generalise inverse semigroups to a certain direction. Instead two new descriptions for semilattices, one for groups, one for E-inversive semigroups and one for completely simple semigroups is found.

1 About this document

Paragraph 3 is part of my bachelor thesis (2002), paragraphs 4 and 5 are parts of my master thesis (2004) (both from University of Tartu, Estonia). Paragraph 4 generalises paragraph 3. In 2007 I solved the open problem that appeared in the master thesis and paragraph 5.4 is partly rewritten accordingly.

2 Regular and inverse semigroups

Here is short introduction to regular and inverse semigroups taken from [Ki]. Let S denote a semigroup wherever not mentioned, and "iff" means "if and only if".

Definition 2.1 An element $a \in S$ is called ***regular*** if there exists $x \in S$ such that $a = axa$; x is called a ***pseudoinverse*** of a. Semigroup S is called ***regular*** if all its elements are regular.

Definition 2.2 An element a' of a semigroup S is called an ***inverse of*** $a \in S$ if $a = aa'a$ and $a' = a'aa'$.

If S is regular and $a = axa$, $a, x \in S$, then it is easy to check that a has an inverse element xax. Also, when a has an inverse a' then it is clearly a pseudoinverse of a. So regular semigroups can be defined as those having an inverse for any of its elements.

Definition 2.3 Semigroup S is called ***inverse*** if its every element has precisely one inverse element.

So if S is inverse, $s \in S$ and $s', s'' \in S$ are inverses of s, then $s'' = s'$.

Theorem 2.4 *A regular semigroup is inverse iff its idempotents commute.*

PROOF. NECESSITY. Let S be an inverse semigroup, $e, f \in S$ two idempotents and $x = (ef)^{-1}$. That means $efxef = ef$ and $xefx = x$. Then $(fxe)(fxe) = f(xefx)e = fxe$, which shows that fxe is an idempotent and so it is an inverse for itself. Also, ef is an inverse for fxe:

$$(ef)(fxe)(ef) = effxeef = efxef = ef$$

and

$$(fxe)(ef)(fxe) = fxeeffxe = fxefxe = fxe.$$

Since S is inverse, $ef = fxe$. Multiplying left by f we get $fef = ffxe = fxe = ef$. Analogously $efe = ef$. Now switching the roles of e and f we get similarly $efe = fe$ and $fef = fe$. So $ef = fe$.

SUFFICIENCY. Let S be a regular semigroup with commuting idempotents and $a \in S$ arbitrary. Suppose both $a', a'' \in S$ are inverses of a. Then

$$a = aa'a,\ a' = a'aa',\ a = aa''a,\ a'' = a''aa''$$

and $aa', aa'', a'a, a''a$ are idempotents. Now we have

$$aa' = (aa''a)a' = (aa'')(aa') = (aa')(aa'') = (aa'a)a'' = aa''$$

because idempotents commute by assumption. Analogously $a'a = a''a$. Finally,

$$a' = a'aa' = (a'a)a' = (a''a)a' = a''(aa') = a''(aa'') = a''aa'' = a''.$$

□

3 k-inverse semigroups: a description for semilattices

Let k be a natural number (that means $k \geq 1$) and S a semigroup. The set of idempotents is denoted by $E(S)$.

Definition 3.1 We call $s \in S$ ***k*-regular**, if there exist $s_1, s_2, \ldots, s_k \in S$ such that

$$s = ss_1s_2 \cdots s_ks.$$

We call $s_1, s_2, \ldots, s_k \in S$ (in that order) ***k*-pseudoinverses** of s. We call S ***k*-regular**, if all its elements are k-regular.

Proposition 3.2 *An element s is k-regular iff it is regular.*

PROOF. NECESSITY. Obvious.

SUFFICIENCY. Notice that if $s = sxs$, then $s = s(xs)^n$ for every $n \in \mathbb{N}$ because $xs \in E(S)$. Case $k = 1$ is trivial. For any $k \geq 2$ we have

$$s = s(xs)(x)s = s(xs)(xs)(x)s = \ldots = s(xs)^{k-1}(x)s.$$

□

Corollary 3.3 *Semigroup S is k-regular iff it is regular.* □

For a regular $s = sxs$ we have two idempotents sx and xs. Next proposition generalises this and will be used later.

Proposition 3.4 *If for some $s, s_1, \dots, s_k \in S$ the equality $s = ss_1s_2 \cdots s_ks$ holds, then $s_1 \cdots s_ks$, $ss_1 \cdots s_k$ and*

$$s_i \cdots s_kss_1 \cdots s_{i-1}, \quad i = 2, 3, \dots, k,$$

are all idempotents.

Proof. The proof is a direct calculation:

$$\begin{aligned}
(s_1 \cdots s_ks)(s_1 \cdots s_ks) &= s_1 \cdots s_k(ss_1 \cdots s_ks) = s_1 \cdots s_ks, \\
(ss_1 \cdots s_k)(ss_1 \cdots s_k) &= (ss_1 \cdots s_ks)s_1 \cdots s_k = ss_1 \cdots s_k,
\end{aligned}$$

$$\begin{aligned}
& (s_i \cdots s_kss_1 \cdots s_{i-1})(s_i \cdots s_kss_1 \cdots s_{i-1}) \\
= \ & s_i \cdots s_k(ss_1 \cdots s_{i-1}s_i \cdots s_ks)s_1 \cdots s_{i-1} \\
= \ & s_i \cdots s_kss_1 \cdots s_{i-1}.
\end{aligned}$$

□

Now we introduce the concept of k-inverse elements.

Definition 3.5 We call $s_1, s_2, \dots, s_k \in S$ (in that order) ***k-inverse*** elements for $s \in S$, if

$$\begin{aligned}
s &= ss_1s_2 \cdots s_ks, \\
s_1 &= s_1s_2 \cdots s_kss_1, \\
s_2 &= s_2s_3 \cdots s_kss_1s_2, \\
\cdots & \cdots \cdots \\
s_{k-2} &= s_{k-2}s_{k-1}s_kss_1 \cdots s_{k-3}s_{k-2}, \\
s_{k-1} &= s_{k-1}s_kss_1 \cdots s_{k-2}s_{k-1}, \\
s_k &= s_kss_1s_2 \cdots s_{k-1}s_k.
\end{aligned}$$

It is clear that the former inverse element is 1-inverse element. We can now generalise the fact that regular element has an inverse element. So this proposition generalises the discussion after Definition 2.2.

Proposition 3.6 *Let $k \in \mathbb{N}$. Every regular element has k-inverse elements.*

Proof. Let $x \in S$ be an inverse of s, that means $s = sxs$ and $x = xsx$. Define $t_1, \dots, t_k$ as follows:

$$\begin{aligned}
& t_1 = x, t_2 = s, t_3 = x, t_4 = s, \dots, t_{2n} = s, t_{2n+1} = x, \ \text{ if } \ k = 2n+1, n \in \mathbb{N}, \\
& t_1 = x, t_2 = s, t_3 = x, t_4 = s, \dots, t_{2n-1} = x, t_{2n} = sx, \ \text{ if } \ k = 2n, n \in \mathbb{N}.
\end{aligned}$$

We show that $t_1, \dots, t_k$ are k-inverse elements of s.

If $k = 2n + 1$, then

$$\begin{aligned}
& st_1t_2 \cdots t_{2n+1}s = sxsxs \cdots xs = s(xs)(xs) \cdots (xs) = sxs = s, \\
& t_1t_2 \cdots t_{2n+1}st_1 = xsxsx \cdots xsx = x(sx)(sx) \cdots (sx) = xsx = x = t_1, \\
& t_2t_3 \cdots t_{2n+1}st_1t_2 = sxsxs \cdots xsxs = s(xs)(xs) \cdots (xs) = sxs = s = t_2, \\
& \cdots \\
& t_{2n+1}st_1t_2t_3 \cdots t_{2n+1} = xsxsx \cdots sx = x(sx)(sx) \cdots (sx) = xsx = x = t_{2n+1}.
\end{aligned}$$

If $k = 2n$, then

$$\begin{aligned}
&st_1t_2\cdots t_{2n}s = sxsxs\cdots sx(sx)s = s(xs)(xs)\cdots(xs) = sxs = s,\\
&t_1t_2\cdots t_{2n}st_1 = xsxsx\cdots sx(sx)sx = x(sx)(sx)\cdots(sx) = xsx = x = t_1,\\
&t_2t_3\cdots t_{2n}st_1t_2 = sxsxs\cdots sx(sx)sxs = s(xs)(xs)\cdots(xs) = sxs = s = t_2,\\
&\cdots\\
&t_{2n-1}t_{2n}st_1\cdots t_{2n-1} = x(sx)sx\cdots sx = x(sx)(sx)\cdots(sx) = xsx = x = t_{2n-1},\\
&t_{2n}st_1\cdots t_{2n} = (sx)sx\cdots x(sx) = (sx)(sx)\cdots(sx) = sx = t_{2n}.
\end{aligned}$$

□

Remark In the last proposition k-inverse elements were defined in the easiest way. One may also use $s = ss_1\cdots s_ks$ (instead of $s = sxs$) and define for every $i \in \mathbf{k}$ (that means $i \in \{1, 2, \ldots, k\}$)

$$t_i = (\prod_{j=i}^{k} s_j)s(\prod_{j=1}^{i} s_j) = s_i\cdots s_kss_1\cdots s_i.$$

Then use induction by k to prove these are k-inverse elements for s.

Corollary 3.7 *Let $k \in \mathbb{N}$. Semigroup S is regular iff every element in S has k-inverse elements.* □

Next we prove a lemma which is useful later. Let s be a regular element and $s_1, \ldots, s_k$ its k-inverse elements. Then $s = ss_1\cdots s_ks$ plus additional k equalities. It follows that $s = ss_1\cdots s_kss_1\cdots s_ks$ and so on, we can iterate that equality as much as needed. If we do that m times (in case $m = 0$ we have the first equality with no iterations) we have

$$s = s(s_1\cdots s_ks)^m s_1\cdots s_ks. \quad (\star)$$

Between the first and the last s there are $(k+1)m + k = mk + m + k$ elements (factors). Next lemma says that if we put brackets in that expression (or split it) (of course, leaving out the first and the last s), so inducing q factors, $1 \leq q \leq mk + m + k$, then they are q-inverse elements for s.

We may assume that no "new" factor itself has more than $k+1$ factors (from the set $\{s_1, \ldots, s_k\}$), because otherwise it can be shortened by using relations shown in Definition 3.5. (If any of $s_1, \ldots, s_k$ might be presented as a product of two or more elements of S, then we don't count these factors here.)

Lemma 3.8 *Let $s_1, \ldots, s_k \in S$ be k-inverse elements for $s \in S$ and let $m \in \mathbb{N}_0 = \{0, 1, 2, ...\}$ be the number of iterations of the equality $s = ss_1\cdots s_ks$. Put brackets in $(\star)$ so that no new factor has more than $k + 1$ factors as explained before this Lemma, let the new factors be $t_1, \ldots, t_q$, $1 \leq q \leq mk + m + k$. Then $t_1, \ldots, t_q$ are q-inverse elements for s.*

PROOF. The first equality in Definition 3.5 holds because of the way we induced q new elements (see $(\star)$). Now let $j \in \mathbf{q}$ be arbitrary. We want to show that

$$t_j\cdots t_qst_1\cdots t_j = t_j.$$

Let $t_j = s_u\cdots s_v$. (If the first or the last factor here is s, we may denote $s = s_0$; in that case $u = 0$ or $v = 0$ (or both, if $t_j = s$).)

Let $u \leq v$. Then by Proposition 3.4 we get

$$\begin{aligned} t_j \cdots t_q s t_1 \cdots t_j &= s_u \cdots s_k s s_1 \cdots s_k s s_1 \cdots \cdots s_k s s_1 \cdots s_v \\ &= s_u \cdots s_k (s s_1 \cdots s_k)^r s s_1 \cdots s_v \\ &\overset{3.4}{=} s_u \cdots s_k (s s_1 \cdots s_k) s s_1 \cdots s_v \\ &= s_u \cdots s_k (s s_1 \cdots s_k s) s_1 \cdots s_v \\ &= s_u \cdots s_k s s_1 \cdots s_v \\ &= s_u \cdots s_v \cdots s_k s s_1 \cdots s_v \\ &= s_u \cdots s_{v-1} (s_v \cdots s_k s s_1 \cdots s_v) \\ &= s_u \cdots s_{v-1} s_v \\ &= t_j, \end{aligned}$$

where the precise value of $r \in \mathbb{N}_0$ is not important. We get the fifth and the eighth equality because $s_1, \ldots, s_k$ are k-inverse for s.

The case $u > v$ is similar. □

Definition 3.9 We call S ***k*-inverse**, if its every element has unique k-inverse elements.

That means, if in a k-inverse semigroup both $s_1, \ldots, s_k$ and $t_1, \ldots, t_k$ are k-inverse elements for s, then $t_i = s_i, i \in \mathbf{k}$. It is clear that inverse semigroup is now 1-inverse.

That's a strong condition and we will prove that k-inverse semigroup is precisely a semilattice for any $k \geq 2$.

Lemma 3.10 *Let S be a semigroup and $s \in S$ a regular element such that its $(k+1)$-inverse elements are unique. Then for that element there exist unique k-inverse elements.*

Proof. First of all, a regular element *has* $(k+1)$-inverse elements by Proposition 3.6. Let $s_1, \ldots, s_{k+1}$ be unique $(k+1)$-inverse elements for s. Then

$$\begin{aligned} &s = s s_1 \cdots s_k s_{k+1} s, \\ &s_i = s_i \cdots s_{k+1} s s_1 \cdots s_i, \quad i \in \mathbf{k+1}. \end{aligned}$$

Define

$$t_1 = s_1,\ t_2 = s_2,\ \ldots,\ t_{k-1} = s_{k-1},\ t_k = s_k s_{k+1}. \quad (\star)$$

Since $t_1, \ldots, t_k$ are obtained by grouping $s_1, \ldots, s_{k+1}$ we can use lemma 3.8 (with $k = k+1$, $q = k$, $m = 0$) by which $t_1, \ldots, t_k$ *are* k-inverse elements for s.

It is left to show that they are unique. Assume $t'_1, \ldots, t'_k$ are also k-inverse elements for s:

$$\begin{aligned} &s = s t'_1 \cdots t'_k s, && (*) \\ &t'_i = t'_i \cdots t'_k s t'_1 \cdots t'_i, \quad i \in \mathbf{k}. && (i) \end{aligned}$$

We want to show that $t'_i = t_i, i \in \mathbf{k}$. Iterating $(*)$ once we have

$$s = s t'_1 \cdots t'_k s = s t'_1 \cdots t'_k (s t'_1 \cdots t'_k) s.$$

By lemma 3.8 ($k = k$, $q = k+1$, $m = 1$) we have that $t'_1, \ldots, t'_k, s t'_1 \cdots t'_k$ are $(k+1)$-inverse elements for s.

So both $s_1, \dots, s_k, s_{k+1}$ and $t'_1, \dots, t'_k, st'_1 \cdots t'_k$ are $(k+1)$-inverse elements for s. By assumption such elements are unique, so

$$s_i = t'_i, i \in \mathbf{k}, \quad \text{and} \quad s_{k+1} = st'_1 \cdots t'_k. \quad (\star\star)$$

Now $t_i \overset{(\star)}{=} s_i \overset{(\star\star)}{=} t'_i$, if $i \in \mathbf{k-1}$, and

$$t_k \overset{(\star)}{=} s_k s_{k+1} \overset{(\star\star)}{=} s_k(st'_1 \cdots t'_k) \overset{(\star\star)}{=} t'_k(st'_1 \cdots t'_k) = t'_k st'_1 \cdots t'_k \overset{(i)_{i=k}}{=} t'_k.$$

That means $t'_i = t_i, i \in \mathbf{k}$. □

As a corollary we have

Proposition 3.11 *Let $k \in \mathbb{N}$. Then $(k+1)$-inverse semigroup is k-inverse.* □

Corollary 3.12 *Let $k \geq 2$. Then k-inverse semigroup is 2-inverse.*

Proof. Proposition 3.11 says that k-inverse semigroup is $(k-1)$-inverse. Using that fact $k-2$ times we have it is 2-inverse. □

Corollary 3.13 *Let $k \in \mathbb{N}$. Then k-inverse semigroup is inverse.*

Proof. Case $k = 1$ is trivial. If $k \geq 2$, then by Corollary 3.12 the semigroup is 2-inverse and now apply Proposition 3.11 with $k = 1$. □

Lemma 3.14 *Let S be semigroup and $s \in S$ a regular element such that its 2-inverse elements are unique, and the first and the third of its 3-inverse elements have also unique 2-inverse elements. Then s is an idempotent.*

Proof. First, s *has* both 2- and 3-inverse elements by Proposition 3.6. Let the 3-inverse elements of s be t_1, t_2, t_3. Then

$$\begin{aligned} s &= st_1t_2t_3s, & (0)\\ t_1 &= t_1t_2t_3st_1, & (1)\\ t_2 &= t_2t_3st_1t_2, & (2)\\ t_3 &= t_3st_1t_2t_3. & (3) \end{aligned}$$

Both t_1, t_2t_3 and t_1t_2, t_3 are 2-inverse for s by lemma 3.8 ($k = 3$, $q = 2$, $m = 0$).

Since t_2, t_3, s are 3-inverse elements for t_1 (see (1)-(3) and (0)), again by lemma 3.8 (with same parameters) both t_2, t_3s and t_2t_3, s are 2-inverse elements for t_1.

Since s, t_1, t_2 are 3-inverse elements for t_3 (see (3) and (0)-(2)), similarly both s, t_1t_2 and st_1, t_2 are 2-inverse elements for t_3.

By assumption 2-inverse elements for s, t_1 and t_3 were unique. Hence

$$t_1 = t_1t_2 \text{ and } t_2t_3 = t_3;$$
$$t_2 = t_2t_3 \text{ and } t_3s = s;$$
$$s = st_1 \text{ and } t_1t_2 = t_2.$$

We need three equalities: $t_1 = t_1t_2$, $t_3s = s$ and $s = st_1$. It follows that

$$s \overset{(0)}{=} st_1t_2t_3s = s(t_1t_2)(t_3s) = st_1s = (st_1)s = ss,$$

that is $s = s^2$. □

Now we've done all the work to deduce the main result of this paragraph.

Theorem 3.15 *Let $k \geq 2$. Then a semigroup S is k-inverse iff it is a semilattice.*

Proof. Necessity. Let S be a k-inverse semigroup and $s \in S$. Then S is naturally k-regular and regular by Corollary 3.3. So s is regular. By Corollary 3.12 S is 2-inverse. Then s has unique 2-inverse elements. By Proposition 3.6 s has 3-inverse elements; they also have unique 2-inverse elements. Now the assumptions of lemma 3.14 are fulfilled and so $s \in E(S)$. Hence S is idempotent semigroup.

Since S is k-inverse it is inverse by Corollary 3.13. Then its idempotents commute (see Theorem 2.4). But S is idempotent semigroup, hence it is commutative. That is, S is semilattice.

Sufficiency. Let S be a semilattice and $s \in S$. S is idempotent semigroup, so k elements $s, \ldots, s$ are k-inverse elements for s. We show that they are unique. Let also $t_1, \ldots, t_k$ be k-inverse elements for s, that is

$$\begin{aligned} s &= st_1 \cdots t_k s, \\ t_i &= t_i \cdots t_k s t_1 \cdots t_i, \quad i \in \mathbf{k}, \end{aligned}$$

and we have to show that $t_i = s, i \in \mathbf{k}$. Indeed, thanks to idempotence and commutativity we have

$$s = st_1 \cdots t_k s = (st_1 \cdots t_k)s = s(st_1 \cdots t_k) = sst_1 \cdots t_k = st_1 \cdots t_k$$

and

$$t_i = t_i \cdots t_k s t_1 \cdots t_i = (t_i \cdots t_k)(st_1 \cdots t_i) = (st_1 \cdots t_i)(t_i \cdots t_k) =$$

$$= st_1 \cdots t_i t_i \cdots t_k = st_1 \cdots t_i \cdots t_k = st_1 \cdots t_k, \quad i \in \mathbf{k},$$

that is $t_i = s, i \in \mathbf{k}$. □

Corollary 3.16 *If $k \geq 2$, then k-inverse semigroup is precisely 2-inverse.*

Corollary 3.17 *If $k, l \geq 2$, then k-inverse semigroup is precisely l-inverse.*

4 Weakly k-inverse semigroups: another description for semilattices

Let's weaken the condition in the definition of k-inverse semigroups: we require the uniqueness up to the equality of the sets, not to the equality piecewise.

Definition 4.1 Let S be a semigroup and $s \in S$ a regular element. We say k-inverse elements $s_1, \ldots, s_k \in S$ of s are ***weakly determined***, if the existence of another collection of k-inverse elements $t_1, \ldots, t_k \in S$ for s implies

$$\{t_1, \ldots, t_k\} = \{s_1, \ldots, s_k\}.$$

Definition 4.2 We call a semigroup ***weakly k-inverse***, if its every element has weakly determined k-inverse elements.

It is clear that a k-inverse semigroup is regular and k-inverse semigroup is weakly k-inverse. That is, for $k \geq 2$, a semilattice is weakly k-inverse due to Theorem 3.15. Now we start again moving closer to the opposite. Let V_a denote the set of all inverse elements of an element a.

Proposition 4.3 *Weakly k-inverse semigroup is inverse for every $k \in \mathbb{N}$.*

PROOF. Let S be a weakly k-inverse semigroup. Case $k = 1$ is trivial. Let $k \geq 2$, $e \in E(S)$ and let $x \in V_e$, that is $e = exe$ and $x = xex$. We split the proof into three parts.

A) We show first that $x \in E(S)$ and even more: $x = e$.

If k is an odd number we may write $e = exexe \cdots xexe = (ex)^{\frac{k+1}{2}}e$. There are $2\frac{k+1}{2} + 1 = k + 2$ elements in that expression and between the first and the last e there are k elements, x and e alternating. By lemma 3.8 they are k-inverse elements for e. Since also k elements e are k-inverse for e, by weakly k-inversity

$$\{x, e, x, e, \dots, x, e, x\} = \{e, e, \dots, e\},$$

or $\{x, e\} = \{e\}$. Hence $x = e$.

If k is even we may write $e = exexe \cdots xexe = (ex)^{\frac{k+2}{2}}e$. There are $2\frac{k+2}{2} + 1 = k + 3$ elements, $k + 1$ between first and last e. To get k elements, we write $e = e(xe)xexe \cdots xexe$. Again by lemma 3.8 elements $xe, x, e, x, e, \dots, x, e, x$ are k-inverse for e. Also $e, \dots, e$ are, so

$$\{xe, x, e, x, e \dots, x, e, x\} = \{e, e, \dots, e\}$$

or $\{xe, x, e\} = \{e\}$ (if $k = 2$, then $\{xe, x\} = \{e\}$). Hence $xe = x = e$.

B) Next we show that the product of two idempotents is an idempotent (that means $E(S)$ is a subsemigroup of S, or: S is orthodox).

Let $e, f \in E(S)$ and $x \in S$ an inverse of ef. As it is shown in necessity part of Theorem 2.4, $fxe \in E(S)$ and $ef \in V_{fxe}$. Now $ef \in E(S)$ by part **A)** (and $ef = fxe$).

C) Finally we show that idempotents commute.

Let $e, f \in E(S)$. Since $ef \in E(S)$,

$$(ef)fe(ef) = efef = (ef)^2 = ef$$

and similarly $(fe)ef(fe) = fe$. That means $fe \in V_{ef}$. By part **A)** an inverse element of an idempotent is equal to that idempotent, that is $fe = ef$. Since S is regular, S is inverse by Theorem 2.4. □

So now we can use the notion s^{-1} instead of more general V_s.

Definition 4.4 Let $k \in \mathbb{N}$. We call s a ***k-idempotent*** if $s = ss^k$. We call a semigroup ***k-idempotent*** if all its elements are k-idempotents.

Usual idempotent is 1-idempotent. That definition is for convenient wording of the

Proposition 4.5 *If $k \geq 3$, then weakly k-inverse semigroup is commutative and 2-idempotent.*

Proof. Let $k \geq 3$, S a weakly k-inverse semigroup, $s \in S$ and $t = s^{-1}$: $s = sts$ and $t = tst$. Then

$$s = s(ts)(ts)\cdots(ts)ts = s(ts)^{k-1}(t)s \quad \text{and}$$

$$s = s(ts)(ts)\cdots(ts)tsts = s(ts)^{k-3}(t)(s)(t)s.$$

By lemma 3.8 and k-inversity $\{ts, t\} = \{ts, t, s\}$ if $k > 3$. If $k = 3$ then this equality is $\{ts, t\} = \{t, s\}$. In both cases $s \in \{ts, t\}$. If $s = ts$ then $s = s(ts) = ss$; if $s = t$ then $s = sts = sss$.

So we have shown that in general S is 2-idempotent for $k \geq 3$. Note also that in this semigroup $s^{-1} = s$ and $s^2 \in E(S)$ for every $s \in S$.

Finally we show that S is commutative. Let $s, t \in S$. Then $(st)^{-1} = st$ and $s^2, t^2 \in E(S)$. Remind that by 4.3 S is inverse and so its idempotents commute (by 2.4). So

$$st(ts)st = st^2s^2t = ss^2t^2t = s^3t^3 = st$$

and similarly $ts(st)ts = ts$. That means ts is also an inverse element for st. Since S is inverse, $ts = st$. □

The assumption $k \geq 3$ was important. We consider the case $k = 2$ at the end of this paragraph. Now we make another step closer to semilattice.

Proposition 4.6 *If $k \geq 3$, then weakly k-inverse semigroup is idempotent semigroup.*

Proof. Let $k \geq 3$, S a weakly k-inverse semigroup and $s \in S$. Depending on the parity of k, the proof splits.

Let $k = 2n + 1$. Then $n \in \mathbb{N}$, because $k \geq 3$. Since

$$s = ss^2 = s(s^2)^{2n+1} = s(s^2)^{2n}ss = ss^2s^2\cdots s^2ss,$$

the set of k-inverse elements for s is $\{s^2, s\}$ (by lemma 3.8). But also

$$s = ss^2 = s(s^2)^{n+1} = ss^{2n+2} = ss^{2n+1}s,$$

so $\{s\}$ is the set of k-inverse elements. Hence by weakly k-inversity $\{s^2, s\} = \{s\}$, that is $s^2 = s$.

Let $k = 2n$. It is clear that k-inverse elements for s^2 as idempotent are k elements all equal to s^2. But also k elements s are k-inverse for s. Indeed, the first equality in Definition 3.5 holds:

$$s^2s\cdots ss^2 = s^2s^{2n}s^2 = s^2(s^2)^ns^2 = s^2s^2s^2 = s^2.$$

For others, denote $s_1 = \ldots = s_{2n} = s$ and choose any $i \in \mathbf{k}$. We have to prove that $s_i \cdots s_{2n}s^2s_1\cdots s_i = s_i = s$. That really is so because

$$s_i\cdots s_{2n}s^2s_1\cdots s_i = s^{2n-i+1}s^2s^i = s^{2n-i+1+2+i} = s^{2n+3} = (s^2)^ns^3 = s^2s = s.$$

By weakly k-inversity $\{s^2\} = \{s\}$, that is $s^2 = s$. □

Remark. As we got the result for k even, we could do the same for k odd ($k = 2n + 1$) by considering k-inverse elements for s^2. Namely, we may write $s^2 = s^2s^2\cdots s^2sss^2$, so the first equality in Definition 3.5 is done if we define

$s_1 = \ldots = s_{2n-1} = s^2$, $s_{2n} = s_{2n+1} = s$. In that case all the other equalities hold:

$$s_i \cdots s_{2n+1}(s^2)s_1 \cdots s_i = s^2 \cdots s^2 ss(s^2)s^2 \cdots s^2 = s^2 \cdots s^2 = s^2 = s_i,$$

if $1 \leq i \leq 2n-1$, and

$$s_{2n}s_{2n+1}s^2 s_1 \cdots s_{2n} = sss^2 s^2 \cdots s^2 s = s^2 s = s,$$
$$s_{2n+1}s^2 s_1 \cdots s_{2n+1} = ss^2 s^2 \cdots s^2 ss = ss^2 = s$$

and the set of k-inverse elements for s^2 are both $\{s^2\}$ and $\{s^2, s\}$.

Now the main result follows.

Theorem 4.7 *If $k \geq 3$, then a semigroup is weakly k-inverse iff it is semilattice.*

PROOF. NECESSITY. Let $k \geq 3$ and S a weakly k-inverse semigroup. By Propositions 4.5 and 4.6 S is commutative idempotent semigroup.

SUFFICIENCY. Is already discussed after Definition 4.2. □

Remark In Proposition 4.5 (which we used in the previous Theorem) we actually didn't need to prove the commutativity. Because alternatively S would be semilattice because it is inverse (Proposition 4.3) and idempotent (Proposition 4.6).

Corollary 4.8 *If $k, l \geq 3$, then S is weakly k-inverse iff it is weakly l-inverse.*□

Corollary 4.9 *If $k \geq 3$, then S is weakly k-inverse iff it is k-inverse.* □

Finally consider the case $k = 2$. We prove that they also are precisely semilattices. Remind that a ***Clifford semigroup*** is a regular semigroup with central idempotents (idempotents commute with all elements). Hence it is inverse. Next Theorem is from book [La], Theorem 5.2.12. As usual, $\mathcal{H}$ is Green relation and μ maximal idempotent separating congruence. We need a definition first.

Definition 4.10 A semigroup S is called ***semilattice of its subsemigroups*** $S_\alpha,\ \alpha \in I$, if $S = \dot{\bigcup}_{\alpha\in I} S_\alpha$ and for every $\alpha, \beta \in I$ there exists unique $\gamma \in I$, such that $S_\alpha S_\beta, S_\beta S_\alpha \subseteq S_\gamma$, that is

$$x \in S_\alpha,\ y \in S_\beta \Rightarrow xy, yx \in S_\gamma.$$

If we look I as a semigroup where the multiplication is defined as $\alpha\beta = \gamma$ (with the notion of previous definition), then I is a semilattice. In other words, we have a lower semilattice I where instead of every $\alpha \in I$ there is S_α. The lower bound of $\alpha, \beta \in I$ is their product $\alpha\beta$. In previous definition, $\gamma = \alpha\beta = \beta\alpha \leq \alpha, \beta$.

Theorem 4.11 *Let S be an inverse semigroup. The following assertions are equivalent:*

(i) S is Clifford semigroup.
(ii) For every $s \in S$, $st = ts$, where t is inverse element of s.
(iii) Every $\mathcal{H}$-class is a group.
(iv) Every μ-class is a group.
(v) S is isomorphic to a strong semilattice of groups.

Proposition 4.12 *Weakly 2-inverse semigroup is precisely a semilattice.*

PROOF. Let S be a weakly 2-inverse semigroup. Then S is inverse by 4.3. We use Theorem 4.11 and show that condition (ii) holds.

Let $s \in S$ and $s = sts$, $t = tst$. Following the proof of Proposition 4.5 we have from $s = st(st)s = s(ts)ts$ that $\{t, st\} = \{ts, t\}$, hence $st \in \{ts, t\}$. If $st = ts$, then we are done. If $st = t$, then $s = (st)s = ts$ and it follows $s = s(ts) = ss$. So s is inverse element to itself, $t = s$ and certainly it commutes with itself. In both cases s commutes with its inverse.

So S is a (strong) semilattice of groups by the same theorem. If we show that every group is trivial, we have that S is semilattice.

Let $S = \dot{\bigcup}_{\alpha \in I} G_\alpha$ and $\alpha \in I$ fixed. Let $1_\alpha \in G_\alpha$ be the identity of G_α and $g \in G_\alpha$ arbitrary. It is clear that $1_\alpha, 1_\alpha$ are 2-inverse elements for 1_α. But also g, g^{-1} are 2-inverse elements for 1_α:

$$\begin{aligned} 1_\alpha &= 1_\alpha g g^{-1} 1_\alpha, \\ g &= g g^{-1} 1_\alpha g, \\ g^{-1} &= g^{-1} 1_\alpha g g^{-1}. \end{aligned}$$

By weakly 2-inversity $\{1_\alpha, 1_\alpha\} = \{g, g^{-1}\}$, that is $1_\alpha = g = g^{-1}$.

SUFFICIENCY. Is explained after Definition 4.2. □

Corollary 4.13 *If $k \geq 2$ then weakly k-inverse semigroup is precisely weakly 2-inverse.* □

Corollary 4.14 *If $k, l \geq 2$ then weakly k-inverse semigroup is precisely weakly l-inverse.* □

Corollary 4.15 *Weakly k-inverse semigroup is precisely k-inverse for every $k \in \mathbb{N}$.* □

5 k-turninversity: descriptions for groups and completely simple semigroups

5.1 Two functions

Let $k \geq 2$ and define a function $f : S \to S^k$ in a regular semigroup so that for every $s \in S$, $f(s) = (s_1, \ldots, s_k)$, where $s_1, \ldots, s_k$ are k-inverse elements for s (in that order). By Proposition 3.6 we know that f is defined everywhere. If we assume f is correctly defined, it means every $s \in S$ has only one sequence of k-inverse elements, that is S is actually k-inverse. And conversely, if S is k-inverse, then f is defined correctly on S. So S is k-inverse precisely when f is defined correctly.

If we take an arbitrary element in a regular semigroup, then we know that it also has an inverse element. But we may also ask – is that arbitrary element an inverse element to some other element? Or pseudoinverse? What if the underlying semigroup is inverse and we ask same questions?

But knowing the concept of k-inverse elements we may ask that if we take ordered (or not ordered) k elements in a semigroup, are they k-inverse elements to some $s \in S$? Or k-pseudoinverse elements? (See definitions 3.1 and 3.5.)

Let's define g, an inverse for f in some sense. We don't require regularity for S. Let $g : S^k \to S$ be a function such that for arbitrary $s_1, \ldots, s_k$, $g(s_1, \ldots, s_k) = s$, where $s_1, \ldots, s_k$ are k-inverse elements for s. If g is correctly defined, then for every ordered sequence $s_1, \ldots, s_k$ there exists precisely one s for which $s_1, \ldots, s_k$ are k-inverse elements. That's how we define new semigroups.

5.2 k-turnregularity and k-turninversity

Let S be a semigroup and $k \in \mathbb{N}$.

Definition 5.1 We call $s_1, \ldots, s_k \in S$ ***k*-turnregular** if there exists $s \in S$ so that $s_1, \ldots, s_k$ are k-pseudoinverse for s (see Definition 3.1): $s = ss_1 \cdots s_k s$. We call a semigroup ***k*-turnregular** if arbitrary k elements in that semigroup are k-turnregular.

Definition 5.2 We call $s_1, \ldots, s_k \in S$ ***k*-turninverse** if there exists $s \in S$ so that $s_1, \ldots, s_k$ are k-inverse elements for s (see Definition 3.5). We call a semigroup ***almost k*-turninverse** if arbitrary k elements in that semigroup are k-turninverse.

Definition 5.3 We call a semigroup S ***k*-turninverse** if for arbitrary k elements $s_1, \ldots, s_k$ there exists an element $s \in S$ for which $s_1, \ldots, s_k$ are k-inverse elements and s is unique.

Uniqueness in previous definition means that if in a k-turninverse semigroup we have k arbitrary elements and there exist s and t for which these k elements are k-inverse, then $t = s$.

For the next proposition we need the concept of E-inversive semigroup. The definition comes from [Mi], where it is mentioned that this semigroup is introduced in 1955 by G. Thierrin.

Definition 5.4 A semigroup S is called ***E*-inversive**, if for every $s \in S$ there exists $x \in S$, such that $sx \in E(S)$.

This definition is symmetric – the side on which the x lies is not important, because taking $y = xsx$, we have $sy, ys \in E(S)$ ([Mi], Lemma 1).

Proposition 5.5 *Let $k \in \mathbb{N}$.*

(i) k-turninverse semigroup is almost k-turninverse;
almost k-turninverse semigroup is k-turnregular.
(ii) Almost k-turninverse semigroup is regular.
(iii) k-turnregular semigroup is precisely E-inversive.
(iv) Almost 1-turninverse semigroup is precisely regular.
(v) 1-turninverse semigroup is precisely inverse.
(vi) Regular semigroup is k-turnregular.
(vii) k-turnregular semigroup is precisely 1-turnregular.

PROOF. (i) Directly from definitions.

(ii) Let S be an almost k-turninverse semigroup and let $s \in S$. Then for k elements $s, \ldots, s$ there exists $t \in S$ so that

$$\begin{array}{ccl} t & = & ts^k t, \\ s & = & s^k ts, \\ s & = & s^{k-1}ts^2, \\ \cdots & \cdots & \cdots \\ s & = & sts^k. \end{array}$$

From the second row we have $s = s(s^{k-1}t)s$ for $k \geq 2$ and directly $s = sts$ for $k = 1$.

(iii) Necessity. Let $s \in S$ and S be a k-turnregular semigroup. Then there exists $x \in S$ so that $x = xs^k x$, hence $s^k x \in E(S)$, that is $s(s^{k-1}x) \in E(S)$ for $k \geq 2$ and $sx \in E(S)$ in the case $k = 1$.

Sufficiency. Let $s_1, \ldots, s_k \in S$, $s = s_1 \cdots s_k$ and S an E-inversive semigroup. Then there exists $x \in S$, so that $sx \in E(S)$. But then $(xsx)s(xsx) = x(sx)^3 = xsx$, that is $(xsx)s_1 \cdots s_k(xsx) = xsx$.

(iv) Necessity is proved in (ii). For sufficiency let S be regular and $s \in S$. That means $s = sxs$ for some $x \in S$. Then $s = s(xsx)s$ and $xsx = (xsx)s(xsx)$, that is xsx is the element for which s is an inverse element.

(v) Necessity. Let S be 1-turninverse semigroup and $s \in S$. Then there exists $t \in S$ so that $t = tst$, $s = sts$ and t is uniquely defined. We want to show that S is inverse. An inverse element for s is t. Is it unique? Let $t' \in S$ be another inverse element for s: $s = st's$ and $t' = t'st'$. Since both t' and t have an inverse element s, from 1-inversity we get $t' = t$. That is, the inverse element of s is uniqely defined.

Sufficiency. Analogous with the necessity.

(vi) and (vii) Follows from (iii). □

So we have the description for k-turnregular semigroups for every $k \in \mathbb{N}$ and descriptions for other two semigroups in case $k = 1$. Examples of E-inversive semigroups can be found in [Mi].

Next we get the description for k-turninverse semigroups.

5.3 The description for k-turninverse semigroups: groups

Noticing the similarity of (iv) and (v), by statement (ii) in Proposition 5.5 we may ask: is k-turninverse semigroup inverse? That is really so.

Proposition 5.6 *k-turninverse semigroup is inverse, $k \in \mathbb{N}$.*

Proof. Let S be a k-turninverse semigroup. If $k = 1$ then we have already proved the statement in Proposition 5.5.(v).

Let $k \geq 2$. Then for k elements s there exists precisely one $t \in S$ so that

$$\begin{array}{ccl} t & = & ts^k t, \\ s & = & s^k ts, \\ s & = & s^{k-1}ts^2, \\ \cdots & \cdots & \cdots \\ s & = & sts^k. \end{array}$$

Since

$$s(s^{k-1}t)s = s^k ts = s$$

and

$$(s^{k-1}t)s(s^{k-1}t) = s^{k-1}(ts^k t) = s^{k-1}t$$

we have $s^{k-1}t \in V_s$. It is left to show that this inverse element is unique. Let $t' \in V_s$: $s = st's$ and $t' = t'st'$. We have to show that $t' = s^{k-1}t$. The rest splits depending on the parity of k.

A) Let k be an even natural number. Define

$$t_1 = s,\ t_2 = t',\ t_3 = s,\ t_4 = t', \ldots, t_{k-1} = s,\ t_k = t's.$$

By lemma 3.8 the elements $t_1, \ldots, t_k$ are k-inverse elements for t' (because $s \in V_{t'}$). We show that $t_1, \ldots, t_k$ are k-inverse elements also for $s^{k-1}t$. In the following we underline the part we change to avoid too much bracketing. First,

$$\begin{array}{rcl} (s^{k-1}t)t_1 \cdots t_k(s^{k-1}t) & = & (s^{k-1}t)\underline{(st')^{r_0}}s(t's)(s^{k-1}t) \\ & = & (s^{k-1}t)\underline{(st')(st's)}(s^{k-1}t) \\ & = & (s^{k-1}t)\underline{s(s^{k-1}t)} \\ & = & s^{k-1}\underline{(ts^k t)} \\ & = & s^{k-1}t \end{array}$$

where the precise value of $r_0 \in \mathbb{N}_0$ is not important (if $k = 2$ then $r_0 = 0$); important is that st' is an idempotent. With this we have proved the validity of the first equality in Definition 3.5.

Now let $i \in \mathbf{k-1}$ be an odd number. Then

$$\begin{array}{rcl} t_i \cdots t_k(s^{k-1}t)t_1 \cdots t_i & = & \underline{(st')^{r_1}}\ \underline{s(t's)}(s^{k-1}t)\underline{(st')^{r_2}}s \\ & = & \underline{(st')s}(s^{k-1}t)\underline{(st')s} \\ & = & \underline{s(s^{k-1}t)s} \\ & = & s^k ts \\ & = & s \\ & = & t_i. \end{array}$$

Precise values of $r_1, r_2 \in \mathbb{N}_0$ are not important (if $i = k-1$ then $r_1 = 0$, if $i = 1$ then $r_2 = 0$).

Let $i \in \mathbf{k-2}$ be an even number. Then

$$\begin{array}{rcl} t_i \cdots t_k(s^{k-1}t)t_1 \cdots t_i & = & \underline{(t's)^{r_3}}(t's)(s^{k-1}t)\underline{(st')^{r_4}} \\ & = & \underline{(t's)(t's)}(s^{k-1}t)(st') \\ & = & \underline{(t's)(s^{k-1}ts)t'} \\ & = & t'(\underline{s^k ts})t' \\ & = & t'st' \\ & = & t' \\ & = & t_i. \end{array}$$

Precise values of $r_3, r_4 \in \mathbb{N}$ are not important (if $i = k-2$ then $r_3 = 1$, if $i = 2$ then $r_4 = 1$).

Finally consider the element t_k. Also in this case (analogously to previous cases so more shortly)

$$\begin{array}{rcl} t_k(s^{k-1}t)t_1 \cdots t_k & = & \underline{(t's)(s^{k-1}t)}\ \underline{(st')^{r_5}s(t's)} \\ & = & t'\underline{(s^k t)s} \\ & = & t's \\ & = & t_k, \end{array}$$

where the precise value of $r_5 \in \mathbb{N}_0$ is not important (if $k = 2$ then $r_5 = 0$).

So we have found k elements, which are k-inverse for both t' and $s^{k-1}t$. Since S is k-turninverse, it follows that $t' = s^{k-1}t$.

B) Let k be an odd natural number. Define

$$t_1 = s,\ t_2 = t',\ t_3 = s,\ t_4 = t', \ldots, t_{k-1} = t',\ t_k = s.$$

These elements are also k-inverse for t' by lemma 3.8. The proof that $t_1, \ldots, t_k$ are k-inverse elements for $s^{k-1}t$ is similar to the part **A)**. So analogously $t' = s^{k-1}t$. □

Remind from inverse semigroup theory that an inverse semigroup is a group iff it has only one idempotent ([La], Proposition 1.4.4). The natural partial order in an inverse semigroup is defined as follows:

$$s \leq t \Leftrightarrow \exists e \in E(S) :\ s = te.$$

Here it doesn't matter on which side of t the idempotent e lies (if $f = tet^{-1}$ where t^{-1} is inverse of t, then $s = te = ft$ – see [La], Lemma 1.4.2). On the set of idempotents $E(S)$ this equality takes the form $e \leq f \Leftrightarrow ef = fe = e$.

Proposition 5.7 *Let S be an inverse semigroup and $\ell \geq 2$. Then the following statements are equivalent:*
(i) S is almost ℓ-turninverse.
(ii) There exists $k \in \mathbb{N},\ k \geq 2$, so that for every $s_1, \ldots, s_k \in S$

$$s_1 \leq s_1 \cdots s_k s' s_1 \quad \text{and} \quad s_k \leq s_k s' s_1 \cdots s_k,$$

where $s' = (s_1 \cdots s_k)^{-1}$.
(iii) S is a group.
(iv) S is almost 2-turninverse.
(v) For every $s_1, s_2 \in S$

$$s_1 \leq s_1 s_2 (s_1 s_2)^{-1} s_1 \quad \text{and} \quad s_2 \leq s_2 (s_1 s_2)^{-1} s_1 s_2.$$

Proof. (i)⇒(ii) We show that $k = \ell$ is what we need. Let S be an almost ℓ-turninverse semigroup and $s_1, \ldots, s_\ell \in S$. Then there exists $s \in S$ so that

$$\begin{array}{rcl} s & = & s s_1 \cdots s_\ell s, \\ s_1 & = & s_1 \cdots s_\ell s s_1, \\ \cdots & \cdots & \cdots \\ s_{\ell-1} & = & s_{\ell-1} s_\ell s s_1 \cdots s_{\ell-1}, \\ s_\ell & = & s_\ell s s_1 \cdots s_\ell. \end{array}$$

By the first equality and by

$$(s_1 \cdots s_\ell) s (s_1 \cdots s_\ell) = (s_1 \cdots s_\ell s s_1) s_2 \cdots s_\ell = s_1 s_2 \cdots s_\ell$$

we have $s \in V_{s_1 \cdots s_\ell}$ (that is s is for s' in the wording of (ii)). The two inequalities now follow from the second and the last equality of the equalities above.

(ii)⇒(iii) Assume (ii) so there is a $k \in \mathbb{N}$ with the given property. Take $s_1 = s_2 = \ldots = s_{k-1} = e \in E(S)$ and $s_k = f \in E(S)$. Then $s_1 \cdots s_k = ef$ and $s' = (s_1 \cdots s_k)^{-1} = (ef)^{-1} = ef$ and

$$s_1 \cdots s_k s' s_1 = ef(ef)e = ef$$

and

$$s_k s' s_1 \cdots s_k = f(ef)ef = ef,$$

because idempotents commute in an inverse semigroup. By assumption $s_1 = e \le ef$ and $s_k = f \le ef$. Since always $ef \le e$ and $ef \le f$, we have $e = ef$ and $f = ef$, that is $e = f$. So there is only one idempotent in this inverse semigroup, so it is a group.

(iii)⇒(i) Let S be a group and $s_1, \dots, s_\ell \in S$. Taking $t = (s_1 \cdots s_\ell)^{-1} = s_\ell^{-1} \cdots s_1^{-1}$ we have $ts_1 \cdots s_\ell t = 1t = t$ and for every $i \in \ell$

$$s_i \cdots s_\ell t s_1 \cdots s_i = (s_i \cdots s_\ell s_\ell^{-1} \cdots s_i^{-1})(s_{i-1}^{-1} \cdots s_1^{-1} s_1 \cdots s_{i-1}) s_i = 11 s_i = s_i.$$

That is, S is almost ℓ-turninverse.

(iv)⇒(v) That is the proof (i)⇒(ii) in the case $\ell = 2$.

(v)⇒(iii) That is the proof (ii)⇒(iii) in the case $k = 2$.

(iii)⇒(iv)That is the proof (iii)⇒(i) in the case $\ell = 2$. □

From the previous Proposition we may deduce that in general a k-turnregular semigroup is not almost k-turninverse. Indeed, if it were, then every inverse semigroup as an example of E-inversive semigroup (which is precisely k-turnregular by 5.5.(iii)) would be group by previous Proposition.

Corollary 5.8 *If $k \ge 2$ then almost k-turninverse semigroup is a group iff its idempotents commute.*

PROOF. NECESSITY. Obvious.

SUFFICIENCY. Almost k-turninverse semigroup is regular by Proposition 5.5.(ii), regular semigroup with commuting idempotents is inverse. Proposition 5.7 says that inverse almost k-turninverse semigroup is a group. □

Theorem 5.9 *If $k \ge 2$ then a semigroup is k-turninverse iff it is a group.*

PROOF. NECESSITY. By Proposition 5.6 a k-turninverse semigroup is inverse and by Proposition 5.5.(i) it is almost k-turninverse. Now by Proposition 5.7 it is a group.

SUFFICIENCY. Let S be a group and $s_1, \dots, s_k \in S$. Then S is almost k-turninverse by the proof (iii)⇒(i) of Proposition 5.7. Now we show that $t = s_k^{-1} \cdots s_1^{-1} \in S$ (from that proof) for which $s_1, \dots, s_k$ are k-inverse elements, is uniquely determined. Assume that $s_1, \dots, s_k$ are k-inverse also for some $s \in S$. Then both s and t are inverses of $s_1 \cdots s_k$ in the group. Consequently $s = t$. □

Corollary 5.10 *If $k \ge 2$ then a k-turninverse semigroup is precisely 2-turninverse.*

Corollary 5.11 *If $k, \ell \ge 2$ then a k-turninverse semigroup is precisely ℓ-turninverse.*

Remind the statement of Theorem 2.4: *A regular semigroup is inverse iff its idempotents commute*. By Proposition 5.5 (iv) and (v) we can say that an almost 1-turninverse semigroup is 1-turninverse iff its idempotents commute. This can be generalised as an interesting analogy.

Corollary 5.12 *For all $k \in \mathbb{N}$ an almost k-turninverse semigroup is k-turninverse iff its idempotents commute.*

PROOF. Case $k = 1$ is clear. Let $k \geq 2$.

NECESSITY. k-turninverse semigroup is group by Theorem 5.9, but in group there's only one idempotent.

SUFFICIENCY. By Corollary 5.8 an almost k-turninverse semigroup with commuting idempotents is a group. The statement now follows using Theorem 5.9. □

5.4 The description for almost k-turninverse semigroups: completely simple semigroups

First we show that almost k-turninverse semigroup cannot be nontrivial semilattice of its subsemigroups.

Proposition 5.13 *Let $k \geq 2$. If almost k-turninverse semigroup is a semilattice of its subsemigroups $S_\alpha, \alpha \in I$, then $|I| = 1$.*

PROOF. Let S be almost k-turninverse and also a semilattice of its subsemigroups: $S = \dot{\bigcup}_{\alpha \in I} S_\alpha$.

Let $\alpha, \beta \in I$ and $\beta \leq \alpha$; then $\alpha\beta = \beta$. Take $s_\alpha \in S_\alpha$ and $s_\beta \in S_\beta$. Since S is almost k-turninverse, there exists $\gamma \in I$ and $s_\gamma \in S_\gamma$ so that $s_\alpha, s_\beta, s_\beta, \ldots, s_\beta$ are k-inverse elements for s_γ. Then $k+1$ equalities hold, from which the second one is $s_\alpha = s_\alpha s_\beta^{k-1} s_\gamma s_\alpha$.

Note that $s_\alpha s_\beta^{k-1} \in S_\alpha S_\beta \subseteq S_{\alpha\beta} = S_\beta$ ($s_\beta^{k-1} \in S_\beta$ because S_β is a subsemigroup). Then

$$s_\alpha = s_\alpha s_\beta^{k-1} s_\gamma s_\alpha \in S_\beta S_\gamma S_\alpha \subseteq S_{\beta\gamma\alpha},$$

which means that $S_\alpha \subseteq S_{\beta\gamma\alpha}$. But we have a direct (ie non-overlapping) union of subsemigroups, so $S_\alpha = S_{\beta\gamma\alpha}$ and $\alpha = \beta\gamma\alpha$, but $\beta\gamma\alpha \leq \beta$. That is $\alpha \leq \beta$ so $\alpha = \beta$. □

Before using this Proposition remind from semigroup theory ([Ho], [Ki]) that a *Rees matrix semigroup without zero* is a set

$$\mathcal{M}(G, A, B, P) = \{(a, g, b) \mid a \in A, g \in G, b \in B\}$$

where G is a group, A and B nonempty sets, $P : B \times A \to G$ a mapping (matrix $|B| \times |A|$ with entries from G) and multiplication is defined as follows:

$$(a, g, b)(c, h, d) = (a, gP_{bc}h, d),$$

where P_{bc} means $P(b, c)$. (See eg [Ki], p. 66.)

It is a well known fact that completely simple semigroups (semigroups that don't have proper ideals and have a primitive idempotent) are precisely Rees matrix semigroups (see [Ho], Theorem 3.3.1).

Proposition 5.14 *A Rees matrix semigroup $S = \mathcal{M}(G, A, B, P)$ is almost k-turninverse, $k \in \mathbb{N}$.*

Proof. Let $(s_1, g_1, t_1), \ldots, (s_k, g_k, t_k) \in S$. We have to define an element for which these elements are k-inverse elements. Let $s \in A, t \in B$ and define

$$\begin{aligned} x &= (P_{ts_1} g_1 P_{t_1 s_2} g_2 P_{t_2 s_3} g_3 \cdots g_{k-1} P_{t_{k-1} s_k} g_k P_{t_k s})^{-1} \\ &= P_{t_k s}^{-1} g_k^{-1} P_{t_{k-1} s_k}^{-1} g_{k-1}^{-1} \cdots g_3^{-1} P_{t_2 s_3}^{-1} g_2^{-1} P_{t_1 s_2}^{-1} g_1^{-1} P_{ts_1}^{-1}. \end{aligned}$$

It turns out that (s, x, t) is what we look for. First,

$$\begin{aligned} & (s, x, t)(s_1, g_1, t_1) \cdots (s_k, g_k, t_k)(s, x, t) \\ = \;& (s, x P_{ts_1} g_1 P_{t_1 s_2} g_2 \cdots P_{t_{k-1} s_k} g_k P_{t_k s} x, t) \\ = \;& (s, 1x, t) \\ = \;& (s, x, t). \end{aligned}$$

Now let $i \in \mathbf{k}$ be fixed. Notice that

$$\begin{aligned} x &= P_{t_k s}^{-1} g_k^{-1} P_{t_{k-1} s_k}^{-1} g_{k-1}^{-1} \cdots g_3^{-1} P_{t_2 s_3}^{-1} g_2^{-1} P_{t_1 s_2}^{-1} g_1^{-1} P_{ts_1}^{-1} \\ &= (P_{t_k s}^{-1} g_k^{-1} \cdots g_{i+1}^{-1} P_{t_i s_{i+1}}^{-1} g_i^{-1})(P_{t_{i-1} s_i}^{-1} g_{i-1}^{-1} \cdots g_1^{-1} P_{ts_1}^{-1}) \\ &= (g_i P_{t_i s_{i+1}} g_{i+1} \cdots g_k P_{t_k s})^{-1} (P_{ts_1} g_1 \cdots g_{i-1} P_{t_{i-1} s_i})^{-1}. \end{aligned}$$

Using this we have

$$\begin{aligned} & (s_i, g_i, t_i) \cdots (s_k, g_k, t_k)(s, x, t)(s_1, g_1, t_1) \cdots (s_i, g_i, t_i) \\ = \;& (s_i, g_i P_{t_i s_{i+1}} g_{i+1} \cdots g_k P_{t_k s} x P_{ts_1} g_1 \cdots g_{i-1} P_{t_{i-1} s_i} g_i, t_i) \\ = \;& (s_i, g_i P_{t_i s_{i+1}} g_{i+1} \cdots g_k P_{t_k s} (g_i P_{t_i s_{i+1}} g_{i+1} \cdots g_k P_{t_k s})^{-1} \cdot \\ & \cdot (P_{ts_1} g_1 \cdots g_{i-1} P_{t_{i-1} s_i})^{-1} P_{ts_1} g_1 \cdots g_{i-1} P_{t_{i-1} s_i} \cdot g_i, t_i) \\ = \;& (s_i, 11 g_i, t_i) \\ = \;& (s_i, g_i, t_i). \end{aligned}$$

□

Note that because $s \in A$ and $t \in B$ were arbitrary, the amount of triples for which fixed $(s_1, g_1, t_1), \ldots, (s_k, g_k, t_k)$ are k-inverse elements, might be big, depending on cardinalities of A and B.

Definition 5.15 A regular semigroup S is called ***completely regular*** if its every element s has an inverse element s' so that $ss' = s's$.

We need a result from [Pe], Theorem (Clifford) II.1.4 (which dates back to 1941).

Theorem 5.16 *The following conditions on a semigroup S are equivalent:*
(i) S is completely regular.
(ii) Every $\mathcal{H}$-class of S is a subgroup.
(iii) S is a union of (disjoint) groups.
(iv) For every $a \in S, a \in aSa^2$.
(v) S is a semilattice of completely simple semigroups.

Theorem 5.17 *Let $k \geq 2$. S is almost k-turninverse semigroup iff it is completely simple.*

Proof. Necessity. Let S be an almost k-turninverse semigroup. First we show that condition of Theorem 5.16.(iv) is satisfied. Let $a \in S$ and $e \in E(S)$. Take k elements $a, a, e, \ldots, e \in S$. By assumption there exists $t \in S$ so that

$$\begin{aligned} t &= ta^2et, \\ a &= a^2eta, \\ a &= aeta^2, \\ e &= eta^2e. \end{aligned}$$

From the third row we get that $a \in aSa^2$.

By the previous theorem, (v) says that S is a semilattice of completely simple semigroups (Rees matrix semigroups). From Proposition 5.13 we get that S is only one such component.

Sufficiency. This is Proposition 5.14. □

This diagram shows what has been proved in this paragraph, $k \geq 2$.

$$\begin{array}{ccccc}
\text{group} & \Rightarrow & \text{completely simple} & & \\
\| & & \| & & \\
k\text{-turninverse} & \Rightarrow & \text{almost } k\text{-turninverse} & \Rightarrow & k\text{-turnregular} \\
\Downarrow & & \Downarrow & & \| \\
1\text{-turninverse} & \Rightarrow & \text{almost 1-turninverse} & \Rightarrow & 1\text{-turnregular} \\
\| & & \| & & \| \\
\text{inverse} & \Rightarrow & \text{regular} & \Rightarrow & E\text{-inversive}
\end{array}$$

Acknowledgement. I am thankful to Mati Kilp, Valdis Laan and Kalle Kaarli for important and improving notes about this work while writing it during my master studies in University of Tartu.

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